

Legendre Expansion and Integral Equations of Displacement Type

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ABSTRACT

Integral equations with an even displacement kernel (which depends only on the difference of the two variables multiplied by a parameter) appear frequently in several applications, e.g. neutron transport theory, electrodynamics and polymer physics. Because of the large range of parameter values of interest, each problem has often led to a separate technique of solution. In this paper, a technique is presented which covers all parameter values. It applies to arbitrary displacement kernels and uses an expansion in Legendre polynomials to give a rapidly converging numerical solution. For small and large parameter values the behaviour of the eigenvalues is theoretically investigated and numerically illustrated. The computational effort involved is relatively very small.

1. INTRODUCTION

The paper discusses a numerical procedure for the determination of the eigenvalues and eigenfunctions of an arbitrary integral equation of the displacement type

$$\lambda\varphi(x) = \int_{-1}^1 K(s|x-y|) \varphi(y) dy, \quad -1 \leq x \leq 1, \quad (1.1)$$

where $K(z)$ is square integrable in $(0, 2s)$ and s is a parameter. Such equations arise in many fields of mathematical physics as well as in purely mathematical contexts. In transport theory [$K(z) = \int_1^\infty \exp(-zt) dt/t$], the neutron flux in slabs and spheres [1-3] is described by an equation of this type. Other fields of application are the theory of signal transmission [4], where $K(z) = \sin z/z$, and the computation of the resistance of a plasma slab between juxtaposed disk electrodes [5, 6] where $K(z) = \int_0^\infty dt \cos zt/(1 + \exp t)$. In polymer physics, the determination of the intrinsic viscosity and friction constant of macromolecules in solution [7, 8] leads to equations with $K(z) = z^{-\xi}$, $0 < \xi < 1$. Several other kernels are discussed in purely mathematical papers [9-12].

The numerical methods described in the literature were usually developed only for the solution of special equations of type (1.1). Two of them should be mentioned here, (a) Legendre expansion of the unknown function [2, 5, 6] and (b) solution of (1.1) in the Fourier representation. The “ j_N -method” of Asaoka [1] belongs to both classes. After application of the Fourier transformation, the kernel of the resultant equation is expanded in spherical Bessel functions. But this is related to the Legendre expansion of the original kernel. Roark and Wing [10] have used the Fourier transform technique to obtain a general method for the solution of (1.1).

Here we shall propose a method which may likewise be applied to any equation of type (1.1). It uses an expansion of the unknown function in an infinite series of Legendre polynomials. Numerical solutions are obtained by truncating the resultant system of algebraic equations at a reasonable order. Although the principle of this method is old [13], it does not seem to have been investigated in detail in connection with arbitrary integral equations of the displacement type.

It will be shown that several properties of the matrix elements and eigenvalues are independent of the actual equation under consideration. Furthermore, we shall find that the Legendre expansion is a powerful tool for the numerical solution of (1.1), because (a) the analytical effort involved is relatively small and (b) the resultant system of algebraic equations can be truncated at a low order. This method converges much faster than that described in [10], thus making earlier truncation possible, and it is applicable with far fewer restrictions. Theoretical considerations and numerical examples confirm this observation. A great portion of the numerical results reported here were obtained by using 4×4 or lower matrices.

2. LEGENDRE EXPANSION

Our main interest lies in the construction of an algorithm suitable for the numerical solution of (1.1). Therefore, no great effort is made to find the minimal conditions under which our results will hold.

We assume in (1.1) that $0 < s < \infty$ (in most cases the method described here applies also to $-\infty < s \leq 0$) and that $K(z)$ is real and in $L_2(0, 2s)$ where $L_2(a, b)$ is the space of functions which are square integrable in (a, b) . It follows from the standard theory of integral equations [13, 14] that

$$K(s | x - y |) \in L_2(-1 \leq x, y \leq 1),$$

and consequently, (1.1) has a discrete spectrum of eigenvalues and the corresponding eigenfunctions are in $L_2(-1, 1)$ [and thus in $L_1(-1, 1)$].

Under these conditions we may expand the solution of (1.1) in an infinite series of Legendre polynomials $P_m(x)$

$$\varphi(x) \sim \sum_{m=0}^{\infty} (2m+1) X_m P_m(x), \quad (2.1)$$

where we have used the sign \sim to state that (2.1) converges in the mean, i.e., for almost all s and x . That is

$$\lim_{N \rightarrow \infty} \int_{-1}^1 \left[\varphi(x) - \sum_{m=0}^N (2m+1) X_m P_m(x) \right]^2 dx = 0. \quad (2.2)$$

Due to the orthogonality of the Legendre polynomials, we get from (2.1) and (1.1), when $\lambda \neq 0$,

$$\begin{aligned} 2X_m &= \int_{-1}^1 \varphi(x) P_m(x) dx \\ &= \lambda^{-1} \int_{-1}^1 dx \int_{-1}^1 dy K(s|x-y|) \varphi(y) P_m(x), \quad m = 0, 1, 2, \dots \end{aligned}$$

Substituting for $\varphi(y)$ the expansion (2.1) and interchanging the sum with the integrals, which is permitted for almost all s under the conditions stated at the beginning of this section, we obtain

$$\lambda X_m = 2 \sum_{n=0}^{\infty} (2n+1) T_{m,n}(s) X_n \quad (2.3)$$

with the matrix elements

$$T_{m,n}(s) = \frac{1}{4} \int_{-1}^1 dx \int_{-1}^1 dy K(s|x-y|) P_m(x) P_n(y), \quad m, n = 0, 1, 2, \dots \quad (2.4)$$

From the latter formula we see that there exists the representation

$$K(s|x-y|) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2m+1)(2n+1) T_{m,n}(s) P_m(x) P_n(y). \quad (2.5)$$

All these manipulations are permitted due to the following theorem:

THEOREM 2.1. *Let $0 < s < \infty$ and $K(z) \in L_2(0, 2s)$. If λ and $\varphi(x)$ are solutions of (1.1), then λ is also a solution of (2.3) and $\varphi(x)$ is approximated in the mean by (2.1). Conversely, if λ and the sequence $\{X_m\}$ are solutions of (2.3), then λ is also a solution of (1.1) and $\varphi(x)$ as defined by (2.1) converges in the mean to the corresponding eigenfunction of (1.1).*

This theorem is given in somewhat more abstract form in [13] and part of it appears explicitly in [10]. For details, we refer to [10, 13, 14] and to the literature cited there.

An approximate solution of (1.1) is obtained by truncating (2.3) at a certain order N

$$\lambda_N X_m^N = 2 \sum_{n=0}^N (2n+1) T_{m,n}(s) X_n^N. \quad (2.6)$$

The eigenfunctions of (2.1) are then approximated by

$$\varphi_N(x) = \sum_{m=0}^N (2m+1) X_m^N P_m(x). \quad (2.7)$$

The justification of this approximation requires another theorem, which again appears in somewhat more abstract form in [13].

THEOREM 2.2. *Suppose $0 < s < \infty$ and $K(z) \in L_2(0, 2s)$. Then λ_N and the sequence $\{X_m^N\}$, which are the solution of (2.6), converge strongly to the solution of (2.3), λ and the sequence $\{X_m\}$, respectively, as $N \rightarrow \infty$.*

From the practical point of view, two items are now left to be demonstrated, (a) that there exists a simple way to evaluate $T_{m,n}(s)$ for arbitrary parameter values and indices and (b) that the truncation order N in (2.6) can be chosen relatively small.

3. THE MATRIX ELEMENTS

The evaluation of (2.4) is facilitated by the fact that one of the integrations can be carried out *independently* of the kernel of (1.1). Since $K(s|x-y|)$ is by definition an even function of $x-y$, we can apply the Fourier cosine integral theorem

$$K(s|x-y|) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \int_0^{2s} dw K(w) \cos s(x-y)t \cos tw. \quad (3.1)$$

The w integration runs, according to the suppositions in Theorem 2.1, only over $(0, 2s)$ thus securing the existence of (3.1) for almost all s, x , and y . Putting (3.1) into (2.4) and substituting $z/2s$ for t and $2sq$ for w , we get

$$\begin{aligned} T_{m,n}(s) = & \frac{1}{4\pi} \int_{-\infty}^{\infty} dz \int_0^1 dq \int_{-1}^1 dx \int_{-1}^1 dy K(2sq) \cos zq \\ & \cdot P_m(x) P_n(y) \left(\cos \frac{zx}{2} \cos \frac{zy}{2} + \sin \frac{zx}{2} \sin \frac{zy}{2} \right), \quad m, n = 0, 1, 2, \dots \end{aligned} \quad (3.2)$$

Application of [15, Sec. 3.32]

$$\int_{-1}^1 P_m(x) \exp\left(\frac{izx}{2}\right) dx = 2i^m j_m\left(\frac{z}{2}\right), \quad m = 0, 1, 2, \dots, \quad (3.3)$$

where $j_m(x)$ are the spherical Bessel functions

$$j_m(x) = (-1)^m j_m(-x) = (\pi/2x)^{1/2} J_{m+1/2}(x), \quad (3.4)$$

leads to

$$T_{m,n}(s) = \begin{cases} \int_0^1 Q_{m,n}(q) K(2sq) dq, & m+n \text{ even,} \\ 0, & m+n \text{ odd.} \end{cases} \quad (3.5)$$

The function

$$Q_{m,n}(q) = \frac{i^{m-n}}{\pi} \int_{-\infty}^{\infty} j_m\left(\frac{z}{2}\right) j_n\left(\frac{z}{2}\right) \cos qz dz \quad (3.6)$$

depends in *no way* on the actual equation to be solved and can be determined once for ever. It is also worth noticing that $Q_{m,n}(q)$ is independent of the parameter s . In addition, it will be necessary to consider only those values of m and n for which $m+n$ is even and this will be understood throughout the paper. This implies that $Q_{m,n}(q)$ is real, as desired.

The integral (3.6) is investigated in detail in another paper [16]. Those properties of $Q_{m,n}(q)$, which are needed here, are listed in Appendix I.

It should be noted that the transformation from (2.4) to (3.5), with Q given by (A-1), can easily be performed, without the need of the Fourier transformation, by the substitution

$$x - y = 2q, \quad x + y = 2z. \quad (3.7)$$

The reason for the use of (3.1) is that it seems to be difficult to obtain (A-4) and (A-5) if (3.7) is taken. On the other hand, in the Fourier representation, (A-4) and (A-5) are verified in a simple way. These identities are essential for the derivation of many other properties of Q .

The transformation from (2.4) to (3.5) has still other interesting consequences. For example, we get for Schmidt's bounds on the eigenvalues

$$\lambda^2 \leq \int_{-1}^1 dx \int_{-1}^1 dy K^2(s|x-y|) = 8 \int_0^1 (1-q) K^2(2sq) dq.$$

It follows from (3.2) that the matrix elements vanish for $m+n$ odd because $P_m(x)$ is either even or odd. Consequently, (2.3) and (2.6) degenerate in two *uncoupled* systems. This leads to the following well known [10] lemma:

LEMMA 3.1. *Any eigenfunction of (1.1) can be assumed to be an even or an odd function.*

The even eigenfunctions $\varphi^+(x)$ are thus approximated by

$$\varphi_M^+(x) = \sum_{m=0}^M (4m+1) X_{2m}^M P_{2m}(x) \quad (3.8)$$

and the associated eigenvalues λ_M^+ satisfy

$$\lambda_M^+ X_{2m}^M = 2 \sum_{n=0}^M (4n+1) T_{2m,2n}(s) X_{2n}^M, \quad m = 0, 1, 2, \dots \quad (3.9)$$

The odd eigenfunctions and the corresponding eigenvalues are similarly approximated by replacing $2m$ and $2n$ by $2m+1$ and $2n+1$ in the above formulae.

Multiplication of (A-4) by $K(2sq)$ and integration over q in $(0, 1)$ gives a linear relation among four matrix elements

$$(2n+1)[T_{m+2,n}(s) - T_{m,n}(s)] = (2m+3)[T_{m+1,n-1}(s) - T_{m+1,n+1}(s)]. \quad (3.10)$$

This relation again is *independent* of the actual equation to be solved. Furthermore, since $Q_{m,n}(q)$ is symmetric with respect to the indices, so also is $T_{m,n}(s)$. Thus, all matrix elements can be determined from (3.10) if one knows those with $n=0$ and $n=m$.

The truncated equation (2.6) requires the determination of $(N+1)^2$ matrix elements. Looking on the special form of Q , we see that only $3(N+1)/2$ integrations are actually necessary. In addition, they can be performed recursively, because the Q 's are polynomials.

Experience has shown that (3.10) can be of little numerical value if the matrix elements are of the same order of magnitude. It is then better to use (3.10) only for the determination of the analytical expression of a matrix element and to extract from that the numerical value for a given s .

Finally, we mention a differential equation which can also be used for the recurrent determination of the matrix elements. Multiplication of (A-5) by $K(2sq)$ and integration over q in $(0, 1)$ gives

$$\begin{aligned} (2n+3) T_{m-1,n+1}(s) + s \frac{d}{ds} [T_{m,n}(s) - T_{m,n+2}(s)] \\ = (n-m-1) T_{m,n}(s) + (m+n+4) T_{m,n+2}(s). \end{aligned} \quad (3.11)$$

There exist six equations of type (3.11) corresponding to the six possible configurations of three neighbouring matrix elements with $m+n$ even. The solutions of these equations contain an integral over previously calculated matrix elements. This is equivalent to the repeated integrations which appear implicitly in (3.5).

4. SMALL PARAMETER VALUES

The determination of the matrix elements can be further simplified if the kernel has the series expansion

$$K(2sq) = \sum_{\sigma=0}^{\infty} a_{\sigma} s^{\sigma} q^{\sigma} = \sum_{\sigma=0}^{\infty} \frac{(\sigma+1)(\sigma+2)}{2} b_{\sigma} s^{\sigma} q^{\sigma}, \quad (4.1)$$

which converges in a certain domain $|s| < R$ around the s origin (q is always ≤ 1). Trivially, it then converges absolutely and uniformly in this domain. Substitution of (4.1) into (3.5) and interchange of sum and integral yields

$$T_{m,n}(s) = \sum_{\sigma=0}^{\infty} b_{\sigma} \psi_{\sigma}^{m,n} s^{\sigma}, \quad (4.2)$$

where

$$\psi_{\sigma}^{m,n} = \frac{(\sigma+1)(\sigma+2)}{2} \int_0^1 q^{\sigma} Q_{m,n}(q) dq \quad (4.3)$$

is given by (A-3). Thus, knowing the expansion of $K(2sq)$, we can immediately write down the expansion of all matrix elements because the expression for $\psi_{\sigma}^{m,n}$ is of great simplicity and depends again in *no way* on the actual equation to be solved.

LEMMA 4.1. *Suppose the expansion (4.1) converges for $|s| < R$. Then (4.2), the expansion of each matrix element, converges absolutely and uniformly for $|s| < R$.*

From (A-3), $|\psi_{\sigma}^{m,n}| \leq 1$ and for $\sigma \geq m+n$, the sign of $\psi_{\sigma}^{m,n}$ does not depend on σ . Therefore the expansion (4.1) of the kernel is a majorant to all matrix elements (4.2).

Formula (4.2) may even be valid if the kernel has no series expansion. An example is the neutron transport kernel $K(2sq) = \int_1^{\infty} dv \exp(-2sqv)/v$ [1-3]. For $s = 0$, this kernel has a logarithmic singularity and so has also the lowest matrix element $T_{0,0}(s)$. But all the other elements are regular at $s = 0$. Knowing the series expansions of $T_{1,1}(s)$ and $T_{2,0}(s)$, which for this kernel are explicitly given in [1], one may in the same way as described above establish the form and the convergence of all other matrix elements.

LEMMA 4.2. *Suppose the series expansions of $T_{m,m}(s)$ and $T_{m+1,m+1}(s)$ converge for $|s| < R$. Then the expansions of all matrix elements $T_{k,l}(s)$ with $k+l \geq 2m$ converge absolutely and uniformly for $|s| < R$.*

This lemma follows immediately from the fact that

$$|\psi_\sigma^{m+k, m-k}| \leq |\psi_\sigma^{m, m}|, \quad k = 1, 2, 3, \dots, m.$$

Another conclusion can still be made from Lemma 4.1. The coefficients of the characteristic polynomial associated with (3.9) are obtained by a finite number of multiplications and additions. Thus we have the following lemma:

LEMMA 4.3. *Suppose the expansion (4.1) converges for $|s| < R$. Then the expansions of the coefficients of the characteristic polynomial of (3.9) converge absolutely and uniformly for $|s| < R$. The same statement holds in the odd case.*

The above results permit us to study the behaviour of the eigenvalues in the neighbourhood of the s origin. Substitution of (4.2) into (3.9) and use of (A-6) and (A-7) shows that the determinantal equation of (3.9) takes on the following form for small values of s :

$$\begin{vmatrix} 2b_0 + O(s) - \lambda & O(s) & O(s^3) & \dots & O(s^{2M-1}) \\ O(s) & O(s) - \lambda & O(s) & \dots & O(s^{2M-3}) \\ O(s^3) & O(s) & O(s) - \lambda & \dots & O(s^{2M-5}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O(s^{2M-1}) & O(s^{2M-3}) & O(s^{2M-5}) & \dots & O(s) - \lambda \end{vmatrix} = 0. \quad (4.4)$$

It follows that

$$\lambda_{M,0}^+ = 2b_0 + O(s), \quad \lambda_{M,k}^+ = O(s), \quad k = 1, 2, 3, \dots, M. \quad (4.5)$$

This result is obviously true for $M = 0$. For $M > 0$, we prove (4.5) by induction. When increasing M by one, the characteristic polynomial associated with (4.4) is multiplied by $\lambda - O(s)$ and (4.5) follows at once. In the same way we obtain for the eigenvalues belonging to the odd eigenfunctions

$$\lambda_{M,k}^- = O(s), \quad k = 0, 1, 2, \dots, M. \quad (4.6)$$

Thus, all eigenvalues, except the extreme one, are proportional to s , but the proportion constants depends on the truncation order M .

Because of Theorems 2.1 and 2.2, we can now pass to the limit $M \rightarrow \infty$ in (4.5) and (4.6) and obtain the following theorem:

THEOREM 4.1. *Suppose $K(2sq)$ has the expansion (4.1). Then, for small s , the eigenvalues of (1.1) behave like*

$$\lambda_0 = 2b_0 + O(s), \quad \lambda_k = O(s), \quad k = 1, 2, 3, \dots \quad (4.7)$$

Finally, we shall treat a special case of (4.1), which appears frequently in the applications [4, 5, 6]. We suppose that the kernel of (1.1), with respect to s , is an even function and has the expansion

$$K(2sq) = \sum_{\sigma=0}^{\infty} a_{2\sigma} s^{2\sigma} q^{2\sigma} = \sum_{\sigma=0}^{\infty} (\sigma + 1)(2\sigma + 1) b_{2\sigma} s^{2\sigma} q^{2\sigma}. \quad (4.8)$$

Substitution of (4.8) into (3.9) and use of (A-6) leads for small s to the following determinantal equation:

$$\begin{vmatrix} O(s^0) - \lambda & O(s^2) & O(s^4) & \cdots & O(s^{2M}) \\ O(s^2) & O(s^4) - \lambda & O(s^6) & \cdots & O(s^{2M+2}) \\ O(s^4) & O(s^6) & O(s^8) - \lambda & \cdots & O(s^{2M+4}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O(s^{2M}) & O(s^{2M+2}) & O(s^{2M+4}) & \cdots & O(s^{4M}) - \lambda \end{vmatrix} = 0. \quad (4.9)$$

For the eigenvalues of (3.9), we then get

$$\lambda_{M,k}^+ = \sum_{\nu=k}^{2M+1} d_{\nu,k}^+ s^{2\nu} + O(s^{4M+4}), \quad k = 0, 1, 2, \dots, M. \quad (4.10)$$

Similarly, in the odd case, we have

$$\lambda_{M,k}^- = \sum_{\nu=k}^{2M+1} d_{\nu,k}^- s^{2\nu+2} + O(s^{4M+6}), \quad k = 0, 1, 2, \dots, M. \quad (4.11)$$

It is an important fact that the coefficients $d_{\nu,k}^{\pm}$ do *not* depend on the truncation order M , and so the series expansion of $\lambda_{M,k}^{\pm}$ differs from the expansion of the corresponding exact eigenvalue of (1.1) by a quantity of the order $O(s^{4M+5\mp 1})$. This can again be proven by induction. When increasing the truncation order in (4.9) from M to $M + 1$, Eq. (A-6) ensures that the coefficient of λ^k in the characteristic polynomial is influenced only by quantities of order $O(s^{2M^2 - 4Mk + 2k^2 + 2M + 2k})$ [$O(s^{2M^2 - 4Mk + k^2 + 4M + 2})$ in the odd case]. We summarize these results by

THEOREM 4.2. *Suppose $K(2sq)$ has the expansion (4.8). Then, for small s , the eigenvalues of (2.6) behave like*

$$\lambda_{N,k} = \sum_{\nu=k}^{N+1} d_{\nu,k} s^{2\nu} + O(s^{2N+4}), \quad k = 0, 1, 2, \dots, N. \quad (4.12)$$

The coefficients $d_{\nu,k}$ vanish for $\nu < k$ and for $k \leq N + 1$, they are equal to the expansion coefficients of the corresponding eigenvalues of (1.1).

The coefficients $d_{v,k}$ can easily be calculated by constructing the characteristic polynomial from (4.12) and comparing the result with the polynomial obtained from (2.6).

5. LARGE PARAMETER VALUES

To study the behaviour of the eigenvalues of (2.6) for large values of the parameter, we change (3.5) by the substitution $z = 2sq$

$$T_{m,n}(s) = \frac{1}{2s} \int_0^{2s} Q_{m,n} \left(\frac{z}{2s} \right) K(z) dz \quad (5.1)$$

and assume

$$\lim_{s \rightarrow \infty} 2 \int_0^{2s} z^m K(z) dz = C_m, \quad (5.2)$$

where C_m is a constant. From (A-2) we have then

$$Q_{m,n} \left(\frac{z}{2s} \right) = \frac{2\delta_{m,n}}{2m+1} - \frac{z}{s} + O(s^{-2}). \quad (5.3)$$

Putting (5.1), (5.2), and (5.3) into (3.9), we have to solve an equation of the form

$$\det \left[O(s^{-2}) - \left(\lambda - \frac{C_o}{s} \right) \delta_{m,n} \right] = 0.$$

Subtracting the $(k+1)$ -th from the k -th row and adding the l -th to the $(l+1)$ -th column, we obtain an equation where the $O(s^{-2})$ dependence is removed except for the last row:

$$\begin{vmatrix} \frac{C_o}{s} + O(s^{-3}) - \lambda & O(s^{-3}) & \cdots & O(s^{-3}) & O(s^{-3}) \\ O(s^{-3}) & \frac{C_o}{s} + O(s^{-3}) - \lambda & \cdots & O(s^{-3}) & O(s^{-3}) \\ \vdots & \vdots & & \vdots & \vdots \\ O(s^{-3}) & O(s^{-3}) & \cdots & \frac{C_o}{s} + O(s^{-3}) - \lambda & O(s^{-3}) \\ O(s^{-2}) & O(s^{-2}) & \cdots & O(s^{-2}) & \frac{C_o}{s} + O(s^{-2}) - \lambda \end{vmatrix} = 0.$$

Performing the same manipulations in the odd case, we obtain

$$\begin{aligned} \lambda_{N,k} &= \frac{C_o}{s} + O(s^{-3}), & k &= 0, 1, 2, \dots, N-2, \\ \lambda_{N,l} &= \frac{C_o}{s} + O(s^{-2}), & l &= N-1, N. \end{aligned} \quad (5.4)$$

From (5.2) and (5.4) we conclude that *all* eigenvalues of (2.6) approach the *same* limit

$$\lim_{s \rightarrow \infty} s\lambda = 2 \int_0^{\infty} K(z) dz. \quad (5.5)$$

This result remains valid under somewhat fewer restrictions. Passing to the limit $N \rightarrow \infty$, we get

THEOREM 5.1. *Suppose, for $k = 1, 2, 3, \dots$*

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^{2s} \left(\frac{z}{2s}\right)^k K(z) dz$$

vanishes with a higher order in s^{-1} than for $k = 0$. Then, as the parameter s tends to infinity, all eigenvalues of (1.1) approach the same limit (5.5).

It should be noted that Theorem 5.1 is a generalization of a well known result of Bellman and Latter [9].

6. NUMERICAL EXAMPLES

The numerical examples given in this section are each chosen to demonstrate special features of the method described in this paper. To provide a comparison, we consider only kernels where it is possible to derive the results also from other sources.

Most of the calculations were performed using 4×4 or lower matrices. The eigenvalue equations of fourth degree were solved by a simple, previously developed, procedure [17]. Generally, the determination in double precision of the eigenvalues belonging to one parameter value required one second on an IBM 7090 computer.

Example 1. The solution of

$$\lambda\varphi(x) = \int_{-1}^1 |x - y| \varphi(y) dy$$

can be obtained in a simple way [12]. Differentiation gives

$$\varphi''(x) - (2/\lambda) \varphi(x) = 0.$$

The eigenvalues belonging to the odd eigenfunctions are

$$\lambda_{2n+1} = -[8/\pi^2(2n + 1)^2]$$

and those belonging to the even eigenfunctions are determined from

$$\sqrt{-\frac{2}{\lambda_{2n}}} \tan \sqrt{-\frac{2}{\lambda_{2n}}} + 1 = 0.$$

This equation has one positive and infinitely many negative roots.

TABLE I
Eigenvalues of $K(z) = z$

N	0	2	4	6	Exact
λ_0	1.33	1.38959	1.3896331	1.3896331	1.3896331
λ_2		-0.247	-0.25535	-0.2553963	-0.2553964
λ_4			-0.043	-0.0528	-0.0534
λ_6				-0.015	-0.023
N	1	3	5	7	Exact
λ_1	-0.80	-0.81056	-0.8105695	-0.8105695	-0.8105695
λ_3		-0.078	-0.08971	-0.0900602	-0.0900633
λ_5			-0.023	-0.0314	-0.0324
λ_7				-0.009	-0.017

Numerical examples up to the truncation order $N = 7$ are given in Table I. We see that already with 3×3 matrices λ_0 and λ_1 can be determined with seven exact digits. When using 4×4 matrices, λ_2 and λ_3 coincide with the correct result up to five digits and λ_4 and λ_5 differ by at most unity in the second digit.

Example 2. Similar good results are obtained from

$$\lambda \varphi(x) = \int_{-1}^1 \exp(-s|x-y|) \varphi(y) dy,$$

where the analytical solution can be obtained in the same way as in Example 1 [3]. The eigenvalues are determined from

$$\lambda_n = 2s/(s^2 + \omega_n^2)$$

and the eigenfrequencies from

$$\begin{aligned} \omega_{2n} \tan \omega_{2n} - s &= 0, \\ s \tan \omega_{2n+1} + \omega_{2n+1} &= 0. \end{aligned}$$

TABLE II
Eigenvalues of $K(z) = \exp(-z)$

s	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
-5.0	2204.414	(46)	-2200.844	-0.34761	-0.25403	-0.178 (9)
-4.0	374.3645	(51)	-370.8650	-0.39655	-0.25528	-0.166 (4)
-3.0	68.88127		-65.54616	-0.43290	-0.23409	-0.140 (3)
-2.0	15.05901		-12.02125	-0.39820	-0.17958	-0.100 (1)
-1.0	4.553435		-2.0000000	-0.226476	-0.094379 (81)	-0.051 (2)
-0.6	3.145972		-0.8172295	-0.133254	-0.056188 (9)	-0.031 (20)
-0.2	2.297902		-0.1913220	-0.042079	-0.018310 (1)	-0.010 (7)
0.2	1.759393		0.1383294	0.038815	0.017664	0.006
0.6	1.400042		0.3082051	0.105403	0.050534 (6)	0.018 (9)
1.0	1.149310		0.3909412	0.157048	0.079552 (7)	0.046 (7)
2.0	0.7752452		0.4329379	0.23154	0.13387 (8)	0.083 (5)
3.0	0.5757076		0.3991966	0.25524	0.16489 (91)	0.109 (11)
4.0	0.4545663		0.3538708	0.25408	0.17907 (11)	0.125 (8)
5.0	0.3741651		0.3120911	0.24230	0.18260 (5)	0.134 (7)
10.0	0.1959984		0.1848506	0.16871	0.15011 (7)	0.128 (31)
20.0	0.09944351		0.09780946 (8)	0.09519	0.09173 (6)	0.086 (8)
50.0	0.03996209		0.03984878	0.03963	0.03940	0.039
100.0	0.01999516		0.01998066 (7)	0.01995	0.01991	0.020

Table II compares our results with the exact ones. Whenever discrepancies occur between the two results, the exact digits differing from our approximation are given in parenthesis. If there are no parenthesis, both results coincide in all digits. We again emphasize that all results in Table II are obtained using 4×4 matrices.

For small s , the eigenvalues show the behaviour indicated in (4.7). It is interesting that λ_4 and λ_5 are proportional to s up to $|s| \sim 1.0$, while for λ_1 , (4.7) holds only for $|s| \leq 0.1$. That is, the higher terms in the expansion of λ play an important role only for the first few eigenvalues.

For large positive s , all eigenvalues tend to $2/s$, as expected from Theorem 5.1. For large negative s , λ_0 and λ_1 behave like $\mp \exp(-2s)/2s$, while all other eigenvalues tend to zero.

Example 3. Consider an equation with an oscillating kernel

$$\lambda\varphi(x) = \int_{-1}^1 \cos s(x-y) \varphi(y) dy. \quad (6.1)$$

The eigenvalues are given by

$$\lambda_0 = 1 + j_0(2s), \quad \lambda_1 = 1 - j_0(2s).$$

The determination of the matrix elements here is simple if one looks at the definition of $Q_{m,n}(q)$, Eq. (3.6). It follows that

$$T_{m,n}(s) = i^{m-n} j_m(s) j_n(s).$$

Since the cos-kernel does not fulfill the requirements of Theorem 5.1, we can expect good results only for small parameter values. Table III shows the size of the $M \times M$ matrices which have to be used to obtain λ_0 and λ_1 exact to seven digits. Since (6.1) has a degenerate kernel, the eigenvalues are simply obtained by summing up the diagonal terms of the respective matrices. The values of λ_2 and λ_3 in our approximation (obtained from 4×4 matrices) are also given to provide a numerical proof of the accuracy of the method. These values should go to zero for infinite truncation order. The negative integers following the λ_2 and λ_3 values denote powers of 10.

The numerical results in Table III show for small s the behaviour predicted by Theorem 4.2. Up to $|s| \sim 0.6$, λ_1 is proportional to s^2 . We further note explicitly that because of the oscillatory behaviour of the kernel, the eigenvalue λ_0 (belonging to the even eigenfunction) can be *smaller* than λ_1 .

Example 4. Finally, consider the equation

$$\lambda\varphi(x) = \int_{-1}^1 J_0[s(x-y)] \varphi(y) dy.$$

TABLE III
Eigenvalues of $K(z) = \cos z$

s	M	λ_0	λ_1	λ_2	λ_3
0.2	2	1.9735459	0.0264541	0	0
0.4	2	1.8966951	0.1033049	0	0
0.6	3	1.7766992	0.2233008	0	0
0.8	3	1.6247335	0.3752665	0	0
1.0	3	1.4546487	0.5453513	0	0
1.5	4	1.0470400	0.9529600	0.13-10	0.64-10
2.0	4	0.8107994	1.1892006	0.19- 7	-0.40- 8
2.5	5	0.8082151	1.1917849	-0.33- 7	-0.87- 8
3.0	5	0.9534308	1.0465692	-0.54- 7	-0.48- 7
3.5	5	1.0938552	0.9061448	-0.23- 6	-0.10- 6
4.0	6	1.1236698	0.8763302	-0.48- 6	-0.19- 6
4.5	6	1.0457909	0.9542091	0.78- 7	0.35- 6
5.0	6	0.9455979	1.0544021	-0.92- 6	-0.51- 6
6.0	7	0.9552856	1.0447144	-0.23- 6	-0.14- 5
7.0	8	1.0707577	0.9292423	0.39- 6	-0.58- 7
8.0	8	0.9820060	1.0179940	-0.27- 6	-0.46- 6
9.0	9	0.9582785	1.0417215	-0.72- 7	-0.65- 6
10.0	10	1.0456473	0.9543527	-0.18- 6	0.54- 8

TABLE IV
Eigenvalues of $K(z) = J_0(z)$

s	λ_0	λ_1	M	λ_2	λ_3
0.2	1.9867375	0.0132536	2	0.8897244	-5
0.6	1.8855726	0.1137001	2	0.7254375	-3
1.0	1.7072416	0.2870747	3	0.5643431	-2
1.5	1.4340351	0.5370153	3	0.2848326	(4)-1
2.0	1.1699387	0.7411874	4	0.862036	(0)-1
2.5	0.9544810	0.8468611	4	0.188333	(24)
3.0	0.7964261	0.8526998	4	0.32020	(14)
4.0	0.6585164	0.7097817	5	0.4821	(17)
5.0	0.6341317	0.5765411	6	0.415	(4)
6.0	0.5416475	0.5533315	7		
8.0	0.4809063	0.4546344	8		

The first four eigenvalues are given in Table IV. For λ_0 and λ_1 , the results are calculated by the ordinary iterative procedure using the Rayleigh quotient. M is again the size of the matrices which must be used to obtain the stated accuracy. For λ_2 and λ_3 , the results are computed from 4×4 matrices and whenever these deviate from those obtained from 3×3 matrices, the differing final digits are given in parenthesis. Since the kernel is an oscillating function, the curves of $\lambda(s)$ also oscillate and consequently, for some ranges of s , $\lambda_1 > \lambda_0$, where λ_0 is again defined as the extreme eigenvalue belonging to an even eigenfunction.

For small s , the numerical values given in Table IV show the behaviour (4.12). More explicitly, λ_1 is proportional to s^2 up to $|s| \sim 0.6$, while λ_3 is proportional to s^6 even as far as $|s| \sim 2.0$. The three digit results for λ_0 and λ_1 of Roark and Wing [10] coincide with our values.

7. FINAL REMARKS

It was shown in the previous sections that the Legendre expansion is a powerful tool for the numerical solution of equations of type (1.1). Naturally, this method may be applied also to more general equations. A simple alteration is to assume in (1.1) arbitrary, but finite, integral boundaries. Infinite boundaries are explicitly excluded because in this case the kernel is not any more square integrable. The connection to the previous formalism is obtained by a simple substitution. If λ_k^* are the eigenvalues, $\varphi_k^*(\xi)$ the eigenfunctions ($k = 0, 1, 2, \dots$) and s^* the parameter of

$$\lambda^* \varphi^*(\xi) = \int_a^b K(s^* | \xi - \eta |) \varphi^*(\eta) d\eta \quad (7.1)$$

and $\lambda_k, \varphi_k(x)$ and s are the corresponding quantities of (1.1), then

$$\begin{aligned} \lambda_k^* &= \rho \lambda_k, & s &= \rho s^*, \\ \varphi_k^*(\xi) &= \varphi_k[(\xi - a)/\rho - 1], \end{aligned} \quad (7.2)$$

where $\rho = (b - a)/2$.

Another possible extension is to assume that the kernel of (1.1) contains two parameters and depends on the argument itself and not on its modulus:

$$K(sx - ry) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2m+1)(2n+1) T_{m,n}(s, r) P_m(x) P_n(y). \quad (7.3)$$

In several cases, depending on the kernel under consideration, the degeneration

of the corresponding eigenvalue matrix in two uncoupled systems will be lost. All properties of Q , which are needed for this case, may be found in [16].

It is also possible, but more complicated, to use other orthogonal polynomials. They should have the property of falling into a class of even and a class of odd functions. This is achieved if the weight function associated with the polynomials is an even function, provided the boundaries of the integral equation are symmetric with respect to the origin [18]. Among the Jacobi polynomials the Gegenbauer polynomials have this property.

A polynomial expansion can also be applied to other than displacement integral equations. One of the main features of such equations is that one of the integrations involved in the determination of the matrix elements can be carried out independently of the actual kernel under consideration. The same applies, at least in principle, to arbitrary integral equations. Another advantage of the method described here is that one can write down the series expansion of all matrix elements if one knows the expansion of the kernel. This holds equally, again at least in principle, for arbitrary integral equations where it is possible to expand the kernel in powers of the parameter.

Finally, we mention that the Legendre expansion can be applied with equal success to inhomogeneous equations of type (1.1). The only change is the appearance of inhomogeneous terms in (2.3), (2.6), and (3.9) respectively, which are the Fourier-Legendre coefficients of the inhomogeneous term of the original equation.

From the numerical point of view, some caution should be observed. A polynomial expansion as described here converges "in the mean." That is, it converges everywhere to the exact solution except in a set of points of measure zero [see Eq. (2.2)]. Therefore, at some points, the solution (2.1) may differ from the exact one. On the other hand, in physical applications, this feature is not usually encountered.

The recurrence formula (3.10) is numerically unstable and should thus be used only for the determination of the analytical expression of a matrix element rather than for the numerical computation. Fortunately, for small parameter values, the elements can be calculated from their series expansion (4.2). In the future it is intended that a similar simple general expression for large parameter values will be found.

APPENDIX I

The integral (3.6) is evaluated in detail in [16]. For $0 \leq q \leq 1$, Q is a special polynomial which is symmetric with respect to the indices, as can be seen from (3.6). For $q \geq 1$, Q vanishes identically. This is equivalent to the finite integration

over w in (3.1). Several other properties of Q are listed below. As usual, empty products are to be replaced by unity.

$$Q_{m,n}(q) = \begin{cases} \int_{-1}^{1-2q} P_m(x) P_n(x+2q) dx, & 0 \leq q \leq 1, \\ 0, & 1 \leq q < \infty; \end{cases} \quad (\text{A-1})$$

$$Q_{m,n}(q) = \sum_{\nu=0}^{m+n+1} c_{\nu}^{m,n} q^{\nu}, \quad c_0^{m,n} = \frac{2\delta_{m,n}}{2m+1},$$

$$c_{\nu}^{m,n} = \frac{2(-1)^{\nu}}{\nu!(\nu-1)!} \prod_{\kappa=1}^{\nu-1} (m+n+1+\nu-2\kappa)(|m-n|+\nu-2\kappa), \\ \nu = 1, 2, 3, \dots, m+n+1; \quad (\text{A-2})$$

$$\frac{(\sigma+1)(\sigma+2)}{2} \int_0^1 q^{\sigma} Q_{m,n}(q) dq \\ = (-1)^m \prod_{\rho=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(\sigma+3-2\rho)}{(\sigma+1+2\rho)} \prod_{\mu=1}^{\frac{m+n}{2}} \frac{(\sigma+2-2\mu)}{(\sigma+2+2\mu)}, \quad \sigma > -1; \quad (\text{A-3})$$

$$(2n+1)[Q_{m+2,n}(q) - Q_{m,n}(q)] = (2m+3)[Q_{m+1,n-1}(q) - Q_{m+1,n+1}(q)]; \quad (\text{A-4})$$

$$(2n+3) Q_{m-1,n+1}(q) - q(d/dq)[Q_{m,n}(q) - Q_{m,n+2}(q)] \\ = (n-m) Q_{m,n}(q) + (m+n+3) Q_{m,n+2}(q); \quad (\text{A-5})$$

$$\int_0^1 q^{2\kappa} Q_{m,n}(q) dq = 0, \quad \kappa = 0, 1, 2, \dots, \frac{m+n-2}{2}, \quad m+n \geq 2; \quad (\text{A-6})$$

$$\int_0^1 q^{\tau} Q_{m,n}(q) dq = 0, \quad \tau = 0, 1, 2, \dots, |m-n|-2, \quad |m-n| \geq 2. \quad (\text{A-7})$$

APPENDIX II

For convenience, the explicit expressions for $Q_{k,k}(q)$ and $Q_{k,0}(q)$ are listed up to $k = 7$.

$$Q_{0,0}(q) = 2(1-q)$$

$$Q_{1,1}(q) = 2(\frac{1}{3} - q + \frac{2}{3}q^3)$$

$$Q_{2,2}(q) = 2(\frac{1}{5} - q + 2q^3 - \frac{6}{5}q^5)$$

$$Q_{3,3}(q) = 2\left(\frac{1}{7} - q + 4q^3 - 6q^5 + \frac{2}{7}q^7\right)$$

$$Q_{4,4}(q) = 2\left(\frac{1}{8} - q + \frac{2}{3}q^3 - 18q^5 + 20q^7 - \frac{7}{9}q^9\right)$$

$$Q_{5,5}(q) = 2\left(\frac{1}{11} - q + 10q^3 - 42q^5 + 80q^7 - 70q^9 + \frac{2}{11}q^{11}\right)$$

$$Q_{6,6}(q) = 2\left(\frac{1}{13} - q + 14q^3 - 84q^5 + 240q^7 - 350q^9 + 252q^{11} - \frac{9}{13}q^{13}\right)$$

$$Q_{7,7}(q) = 2\left(\frac{1}{15} - q + \frac{5}{3}q^3 - \frac{7}{5}q^5 + 600q^7 - \frac{38}{3}q^9 + 1512q^{11} - 924q^{13} + \frac{1}{5}q^{15}\right)$$

$$Q_{2,0}(q) = 2(-q + 3q^2 - 2q^3)$$

$$Q_{4,0}(q) = 2(-q + 10q^2 - 30q^3 + 35q^4 - 14q^5)$$

$$Q_{6,0}(q) = 2(-q + 21q^2 - 140q^3 + 420q^4 - 630q^5 + 462q^6 - 132q^7)$$

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